Name of College - S.S. College, J.Bad

Deptt - Mathematics

Topic - Radius of Curvature (Cont-d)
(Differential calculus)

Class - B.Sc I (Math Hons)

Time - 10.15 A.M to 11 A.M
11 A.M to 11.45 A.M

Date - 29-07-2020

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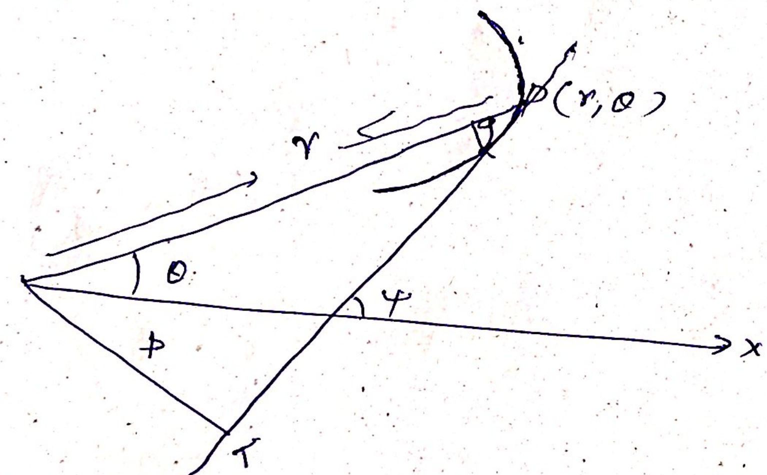
Pedal Formula :→



Let $p = f(r)$ be the pedal Equation of curve.

Let $P(r, \theta)$ be any point on the curve

then $\angle POX = \theta$

From P, a Tangent PT has been drawn to the curve which makes an angle ϕ with OP and angle ψ with the +ve direction of OX.

$$\psi = \theta + \phi$$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \quad \text{- - - -} \quad (1)$$

From O, $OT = p$ has been drawn perpendicular on PT.

Therefore $p = r\sin\varphi$

Diff. w.r.t. respect to r

$$\frac{dp}{dr} = r\cos\varphi\frac{d\varphi}{dr} + \sin\varphi$$

$$= r.\frac{dr}{ds}.\frac{d\varphi}{dr} + r\frac{d\theta}{ds}$$

$$\left[\text{Since } \cos\varphi = \frac{dr}{ds},\quad \sin\varphi = r\frac{d\theta}{ds}\right]$$

$$= r\frac{d\varphi}{ds} + r\frac{d\theta}{ds}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\varphi}{ds}\right)$$

$$= r.\frac{d\psi}{ds}$$

$$= r.\frac{1}{\rho}$$

i.e. $\dfrac{dp}{dr} = r.\dfrac{1}{\rho}$

$$\Rightarrow \rho = r\frac{dr}{dp}$$

(ii) Polar Tangential formula:→

When the equation of curve be $p = f(\psi)$

then $$\frac{dp}{d\psi} = \frac{dp}{dr}.\frac{dr}{ds}.\frac{ds}{d\psi}$$

$$= \frac{dp}{dr}.\cos\varphi.\rho$$

$$\because \frac{dr}{ds} = \cos\varphi \quad \frac{ds}{d\psi} = \rho$$

$$= \cancel{\frac{dp}{dr}\frac{dr}{ds}}.\frac{dp}{dr}\cos\varphi\, r.\frac{dr}{dp}$$

$$= r\cos\varphi$$

Now $p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2\sin^2\phi + r^2\cos^2\phi$

$= r^2$

Diff. both sides with respect to p we get-

$$2p + \frac{d}{dp}\left(\frac{dp}{d\psi}\right)^2 = 2r\frac{dr}{dp}$$

$$\Rightarrow 2p + 2\frac{dp}{d\psi}\,\frac{d^2p}{d\psi^2}\cdot\frac{d\psi}{dp} = 2r\frac{dr}{dp}$$

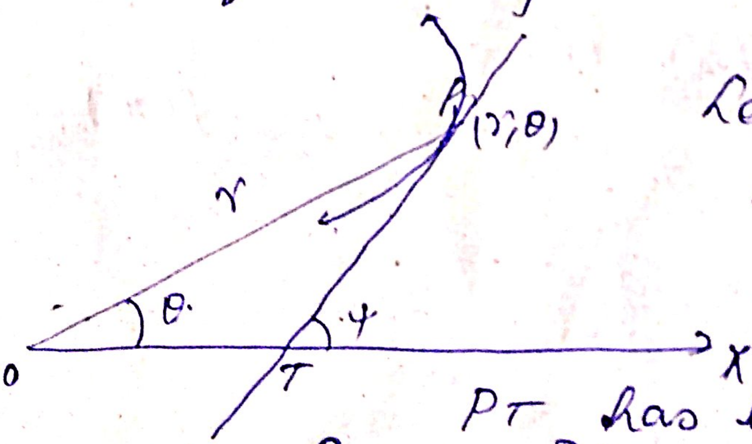
$$\Rightarrow \cancel{2p+2}\; p + \frac{d^2p}{d\psi^2} = r\frac{dr}{dp}$$

$$\because r\frac{dr}{dp} = \rho$$

$$\Rightarrow p + \frac{d^2p}{d\psi^2} = \rho$$

Thus $\rho = p + \frac{d^2p}{d\psi^2}$

Radius of curvature when the equation of curve is $r = f(\theta)$



Let $P(r,\theta)$ be any point on the curve $r = f(\theta)$. From P a tangent PT has been drawn to the curve which makes angle ϕ with radius vector OP and angle ψ with OX.

then $\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$

$= r^2 + r_1^2$

$\frac{ds}{d\theta} = \sqrt{r^2 + r_1^2}$

But since $\tan\phi = r\frac{d\theta}{dr}$

$= \frac{r}{\frac{dr}{d\theta}} = \frac{r}{r_1}$

Therefore Diff with respect to θ, we get

$\sec^2\phi \frac{d\phi}{d\theta} = \frac{r_1 \cdot \frac{dr}{d\theta} - r \cdot \frac{dr_1}{d\theta}}{r_1^2}$

$\sec^2\phi \frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2}$

$(1+\tan^2\phi)\frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2}$

$\left(1 + \frac{r^2}{r_1^2}\right)\frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2}$

$\frac{r_1^2 + r^2}{r_1^2}\frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2}$

$\frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2 + r^2}$

Since $\psi = \theta + \phi$

~~dr~~ $\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$.

$$\frac{d\psi}{d\theta} = 1 + \frac{r_1^2 - r r_2}{r^2 + r_1^2}$$

$$= \frac{r^2 + r_1^2 + r_1^2 - r r_2}{r^2 + r_1^2}$$

But $\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta} \Big/ \frac{d\psi}{d\theta}$

$$= \sqrt{r^2 + r_1^2} \; \frac{r^2 + r_1^2}{r^2 + 2r_1^2 - r r_2}$$

$$= \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

~~Problem~~ → Find the radius of curvature at any point of the curve $r = ae^{m\theta}$

Solution → Here equation of curve is of the form $r = f(\theta)$

then radius of curvature

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

Here $r = ae^{m\theta}$

$r_1 = \frac{dr}{d\theta} = ame^{m\theta} \Rightarrow r_1^2 = a^2 m^2 e^{2m\theta}$

$r_2 = am^2 e^{m\theta}$

$$\therefore \rho = \frac{\left[a^2 (e^{m\theta})^2 + a^2 m^2 (e^{m\theta})^2\right]^{3/2}}{r^2 + 2a^2 m^2 (e^{m\theta})^2 - ae^{m\theta} \cdot am^2 e^{m\theta}}$$

$$= \frac{[r^2 + m^2 r^2]^{3/2}}{r^2 + 2r^2m^2 - m^2r^2}$$

Put $r = ae^{m\theta}$
$r^2 = a^2e^{m\theta, 2}$

$$= \frac{[r^2(1+m^2)]^{3/2}}{r^2[1+2m^2-m^2)}$$

$$= \frac{r^3}{r^2} \frac{(1+m^2)^{3/2}}{1+m^2}$$

$$= r(1+m^2)^{1/2} = r\sqrt{1+m^2}$$

Problem 2

Find the radius of Curvature at any point of the curve.

$$r = a(1+\cos\theta)$$

Solution :→ Here Equation of the Curve is of the form

$$r = f(\theta)$$

∴ By formula

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} \quad \text{——— (1)}$$

Here By the question Equation of the curve

$$r = a(1+\cos\theta)$$

$$r_1 = \frac{dr}{d\theta} = -a\sin\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = -a\cos\theta$$

Putting all these values in (1)

$$\rho = \frac{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta - a(1+\cos\theta)(-a\cos\theta)}$$

$$\Rightarrow \rho = \frac{\left[a^2\left\{(1+\cos\theta)^2 + \sin^2\theta\right\}\right]^{3/2}}{a^2\left\{(1+\cos\theta)^2 + 2\sin^2\theta + \cos\theta(1+\cos\theta)\right\}}$$

$$= \frac{a^3\left\{1+\cos^2\theta + 2\cos\theta + \sin^2\theta\right\}^{3/2}}{a^2\left\{1+2\cos\theta+\cos^2\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta\right\}}$$

$$= \frac{a^3\left\{2 + 2\cos\theta\right\}^{3/2}}{a^2\left\{1 + 3\cos\theta + 2\right\}}$$

$$= \frac{a^3\left\{2(1+\cos\theta)\right\}^{3/2}}{a^2\left[3(1+\cos\theta)\right]}$$

$$= \frac{a^3 \times 2^{3/2} \cdot (1+\cos\theta)^{3/2}}{a^2 \times 3\,(1+\cos\theta)}$$

$$= \frac{a \cdot 2^{3/2} \cdot (1+\cos\theta)^{1/2}}{3}$$

$$= a \cdot 2^{3/2}(2\cos^2\theta/2)^{1/2}$$

$$= a \cdot \frac{2^{3/2} \cdot 2^{1/2}}{3}\cos\theta/2$$

$$= \frac{a \cdot 2^2}{3} \cdot \cos\theta/2$$

$$= \frac{4a}{3}\cos\theta/2$$

Also $\rho = \frac{4a}{3}\cos\theta/2$

$$\Rightarrow \rho^2 = \frac{16}{9}a^2\cos^2\theta/2$$

$$= \frac{\cancel{16}\,8}{9}\,\frac{a^2(1+\cos\theta)}{\cancel{2}}$$

$$= \frac{8}{9}a^2(1+\cos\theta) = \frac{8}{9}ar$$

$$\therefore \rho = 2\sqrt{2ar}$$